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MOTIONS OF A FLUID DROP NEAR A DEFORMABLE INTERFACE

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Abstract—Analytical results are presented for the motion of a viscous Newtonian fluid drop in the presence of a plane, deformable, interface in the velocity range for which inertial effects may be neglected. The zeroth-order approximation for a *spherical* drop near a *flat* interface is expressed in terms of fundamental singularity solutions for Stokes flow, and used to evaluate the drag on the fluid drop in translation either perpendicular or parallel to the interface. The present approximate results for drag are in good agreement with exact-solution results where available. The first corrections for the shapes of the plane interface and the drop are then determined by reformulating the small deformation problem in terms of equivalent boundary conditions on a *flat* interface and a *spherical* drop surface. We consider the influence of the viscosity ratios, density differences and interfacial tensions (or Bond number and capillary numbers) and the drop position relative to the interface, in determining the degree of distortion of the plane interface and the fluid drop surface, and the hydrodynamic drag force on the drop. Among the most interesting results is the prediction of lateral migration induced by the drop and the interface deformations.

Key Words: Stokes flow, domain perturbation, drop/interface deformations, lateral migration

1. INTRODUCTION

There are many processes in chemical and other branches of engineering that involve liquid–liquid or gas–liquid contacting. In many of these cases, the stability of a suspension of drops is of primary importance—namely, the ability of the drops to resist coalescence when brought into close proximity via a mean flow or by Brownian diffusion. Frequently, the size of the drops involved in a close encounter will be comparable, and it is then necessary to solve two- or multi-body fluid dynamics problems to understand the interaction process. In other situations, however, we may be concerned with the relative motions, either convective or diffusive, of two drops which differ greatly in size, or with the motion of a drop near an "unbounded" fluid interface, as in the final stages of the gravity-driven separation of an emulsion into two bulk fluids when each of the fluids still contains finite droplets of the other.

As a logical problem for initial investigation of the complicated phenomena inherent in these applications, we consider a single *fluid* droplet moving near a *deformable* plane interface through a quiescent fluid. When a fluid droplet moves in the vicinity of an interface between two immiscible fluids, the presence of the interface will affect the drop motion, and the interface and the drop will in turn be deformed by the disturbance flow caused by the drop. Three main lines of attack have proved fruitful in studying three-dimensional drop deformation in unbounded fluids: the first, originating with Taylor (1932, 1934), applies when the distortion from sphericity is slight and the familiar technique of domain perturbations can be applied to easily obtain a first approximation to the deformed drope shape (cf. Rumscheidt & Mason 1961a, b; Cox 1969; Frankel & Acrivos 1970; Torza *et al.* 1972; Barthes-Biesel & Acrivos 1973a, b; Rallison 1980); the second, also suggested by Taylor (1964), uses the method of slender-body theory to examine the case where the drop is pulled into a thin thread (cf. Buckmaster 1972; Acrivos & Lo 1978; Hinch & Acrivos 1979, 1980; Brady & Acrivos 1982); and third, numerical techniques have been devised to bridge the gap. Rather than solve for the fluid velocity at all points in space, most studies, following Youngren & Acrivos (1975, 1976), have used a boundary-integral method to cast the creeping-flow equations into an integral form that involves only quantities evaluated on the drop surface (cf. Ladyzhenskaya 1969; Rallison & Acrivos 1978; Rallison 1981).

We have previously considered the motion of a *rigid* particle in creeping motion near a plane *deformable* fluid–fluid interface (Berdan & Leal 1982; Lee & Leal 1982; Geller *et al.* 1986). Berdan & Leal (1982) used the domain perturbation technique to study small deformations of an initially flat interface, with the qualitatively important discovery of a previously unknown form of "lateral migration" induced by the *interface* deformation. It should be noted that the situation is somewhat analogous to the well-known lateral migration of a drop in shear flow away from a plane wall due to deformation of the drop shape (cf. Chaffey *et al.* 1965; Chan & Leal 1979). Furthermore, there is almost certainly an additional lateral velocity component, associated with *sedimentation* of a drop, away from a vertical plane boundary due, again, to shape deformation of the drop. Lee & Leal (1982) and Geller *et al.* (1986) used a boundary-integral technique similar to that of Acrivos and coworkers to consider the problem of a *rigid* sphere in translation *normal* to the interface, without making any *a priori* assumption about the magnitude of the interface deformation.

For the coalescence phenomena and other applications cited earlier, the problem of interest is the motion of a *deformable* drop in the vicinity of a *deformable* interface. Relatively little theoretical work has been done on this problem. In particular, Bart (1968) obtained an exact analytical solution for the drag force on a small *spherical* drop settling toward a *flat* interface. More recently, two studies have been reported from our group using the boundary-integral technique. Chi & Leal (1989) considered the buoyancy-driven translation of a deformable drop normal to a deformable interface, while Ascoli *et al.* (1990) obtained solutions for the approach of a drop to a rigid, plane wall. In both of the latter studies, the magnitude of deformation was not limited in any *a priori* way. However, it is difficult from numerical solutions alone to obtain an understanding of the role of all of the dimensionless parameters for a problem such as the general drop/interface problem which involves three fluids, all of which may be different, and thus two interfaces each characterized by a different interfacial tension. For example, the study of Chi & Leal (1989) only encompassed the special case of a drop of fluid 1, moving through a second fluid toward an interface with its homophase, where there is only a single interfacial tension and a single viscosity ratio and a single density ratio. The present paper represents a complementary investigation in which we obtain *analytic* results for the *general* three-fluid problem, but only for limiting cases where the deformation of the drop and the interface are both small. In this limit, both motions normal and tangential to the interface are of interest. The motions of a particle or drop due to an arbitrarily directed force can be determined in this limit, via *superposition* of results for motions parallel and perpendicular to the interface. Further, the Brownian mobilities of a drop near an interface require knowledge of the hydrodynamic resistance for the parallel and perpendicular motions. The analysis is formally carried out by the method of domain perturbations, in combination with the reciprocal theorem for Stokes flow. The primary thrust of our present research lies in: a systematic assessment of the coexisting role of the drop and interface deformations on the lateral migration of a drop; and an investigation of the effects of the viscosity ratios, the capillary numbers and the Bond number on the drag force and on the distortions of the plane interface and the drop.

2. FORMULATION OF THE PROBLEM

2.1. Governing equations and boundary conditions

We begin by considering the governing equations and boundary conditions for a *fluid* droplet which moves in the vicinity of a fluid interface. The fluid interface and the drop surface separate three immiscible Newtonian fluids that we denote as fluids 1, 2 and 3. The interfaces between fluids 1 and 3 and fluids 2 and 3 are both clean, mobile and characterized completely by constant interfacial tension, denoted respectively as γ_{12} and γ_{23}. The two continuous fluid phases are assumed to be quiescent except for the disturbance flow generated by the motion of the droplet. Further, the undisturbed interface at $z = d$ is assumed to be flat, and the drop is supposed to be wholly immersed in fluid 2. Figure 1 shows a schematic view of the system for a translating fluid droplet near a fluid interface.

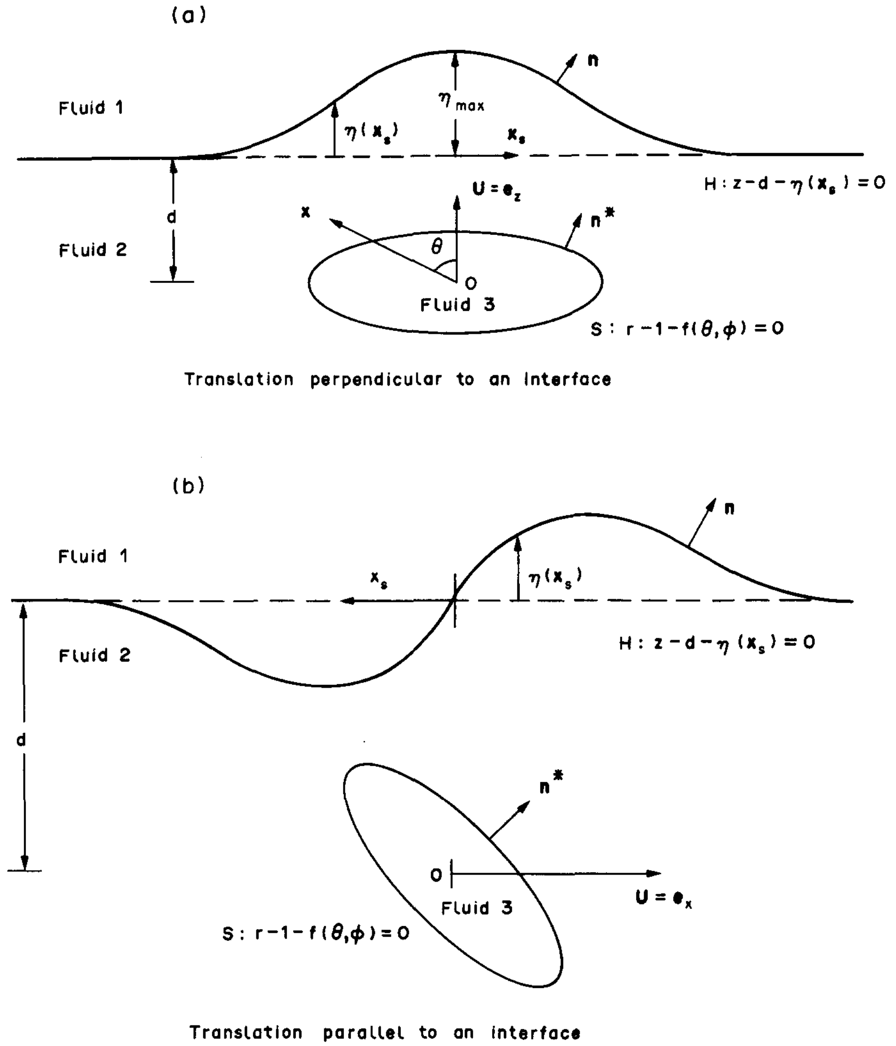

Figure 1. Schematic sketch of the problem. The instantaneous coordinate of the drop center is $\mathbf{x} = \mathbf{0}$ and the undeformed plane interface is represented by $z = d$.

The analysis which we consider is predicated on the neglect of inertia effects in all three fluids. Thus, we assume that the appropriate Reynolds number is sufficiently small, i.e.

$$\mathrm{Re} = \frac{Ua}{\nu_2} \ll 1, \tag{1}$$

where U is the translational velocity of the drop and ν_2 is the kinematic viscosity of fluid 2, and, in addition, that the ratios of kinematic viscosity, ν_1/ν_2 and ν_3/ν_2, are both $O(1)$. The separation distance, d, between the drop center and the undisturbed interface is nondimensionalized by the radius a of the undeformed drop.

The appropriate governing equations thus reduce to Stokes' equation and the equation of continuity in each fluid (see figure 1), i.e. in dimensionless form,

$$\nabla \cdot \mathbf{u} = 0, \qquad \nabla \cdot \boldsymbol{\sigma} = \mathbf{0}, \tag{2}$$

with the stress $\boldsymbol{\sigma}$ and pressure p given by

$$\boldsymbol{\sigma}_i = -p_i\mathbf{I} + \frac{\mu_i}{\mu_2}(\nabla\mathbf{u}_i + \nabla\mathbf{u}_i^{\mathrm{T}}) \qquad (i = 1, 2 \text{ and } 3), \tag{3}$$

in which μ_i is the viscosity of fluid i and $\mathbf{u}_i$ denotes the velocity field in fluid i. The characteristic variables used in the nondimensionalization of [2] and [3] are $u_c = U$, $l_c = a$ and $p_c = \mu_2 U/a$.

It is convenient for the analysis which follows to choose a moving coordinate system in which the drop is at rest with its center of mass at the origin. In the moving frame of reference, a uniform streaming flow, $\mathbf{U}^{\infty} = -\mathbf{U}$, in the creeping-flow limit, is precisely equivalent to translation of the drop with velocity $\mathbf{U}$ relative to a fixed frame of reference in a quiescent fluid. The boundary conditions in the moving frame of reference are

$$\mathbf{u}_1, \mathbf{u}_2 \rightarrow -\mathbf{U} \quad \text{as } |\mathbf{x}| \rightarrow \infty \qquad [4]$$

and, at the drop surface S, defined by $S: r - 1 - f(\theta, \phi, t) = 0$ using spherical polar coordinates (r, θ, ϕ),

$$[|\mathbf{u}^*|]_S = \mathbf{0}, \quad \mathbf{n}^* \cdot \mathbf{u}_2 = \mathbf{n}^* \cdot \mathbf{u}_3 = \frac{1}{|\nabla S|} \cdot \frac{\partial f}{\partial t} \qquad [5a, b]$$

and

$$[|\boldsymbol{\sigma} \cdot \mathbf{n}^*|]_S = \frac{1}{\mathrm{Ca}^*} (\nabla \cdot \mathbf{n}^*)\, \mathbf{n}^* + \frac{1}{\mathrm{Cg}^*} (1 + f) \cos\theta\, \mathbf{n}^*. \qquad [6]$$

The symbol $[|\ |]$ in [5] and [6] represents the jump across the surface of the drop S from the outside to the inside, $\mathbf{n}^*(= \nabla S/|\nabla S|)$ is the outward normal and $\nabla \cdot \mathbf{n}^*$ is the surface curvature. The dimensionless parameter Ca* is the capillary number, $\mathrm{Ca}^* \equiv \mu_2 U/\gamma_{23}$, and Cg* is the body force parameter, $\mathrm{Cg}^* \equiv \mu_2 U/ga^2(\rho_3 - \rho_2)$. The body force parameter appears in [6], because the equations of motion are written in terms of the dynamic pressure; i.e. the stress $\boldsymbol{\sigma}$ in [6] is the total stress minus the hydrostatic pressure contribution. Equation [5b] is the kinematic condition which relates the time rate of change of the drop shape function, $f(\theta, \phi, t)$, to the normal velocities at the drop surface, while [6] is the condition of continuity of stress. At the plane interface, represented by $H: z - d - \eta(\mathbf{x}_s, t) = 0$, we require

$$[|\mathbf{u}|]_H = \mathbf{0}, \quad \mathbf{n} \cdot (\mathbf{u}_1 + \mathbf{U}) = \mathbf{n} \cdot (\mathbf{u}_2 + \mathbf{U}) = \frac{1}{|\nabla H|} \cdot \frac{\partial \eta}{\partial t} \qquad [7a, b]$$

and

$$[|\boldsymbol{\sigma} \cdot \mathbf{n}|]_H = \frac{1}{\mathrm{Ca}} (\nabla \cdot \mathbf{n})\, \mathbf{n} + \frac{1}{\mathrm{Cg}} \eta\, \mathbf{n}. \qquad [8]$$

The parameters appearing in [7] and [8] are the unit outward pointing normal vector $\mathbf{n}$ from fluid 2 (i.e. $\mathbf{n} = \nabla H/|\nabla H|$), and the position vector $\mathbf{x}_s$ representing points lying in a plane parallel to the undeformed, flat, interface at $z = d$. In our model system, the shape function $\eta(\mathbf{x}_s, t)$ is envisioned as distortion from the flat interface, $\eta(\mathbf{x}_s, t) = 0$, due to the disturbance flow. Further, Ca and Cg are dimensionless parameters defined by

$$\mathrm{Ca} = \frac{\mu_2 U}{\gamma_{12}} \quad \text{and} \quad \mathrm{Cg} = \frac{\mathrm{Ca}}{\Phi} = \frac{\mu_2 U}{ga^2(\rho_2 - \rho_1)},$$

which are known, respectively, as the capillary number and the ratio of the capillary number and the Bond number Φ. It can be seen in [8] that the normal component of the stress difference at the interface is balanced by both an interfacial tension force and a buoyancy force, owing to the density difference between the two fluids 1 and 2.

2.2. Solution methodology

The problem represented by [1]–[8] is, of course, both time-dependent and highly nonlinear, due to the fact that $f(\theta, \phi, t)$ and $\eta(\mathbf{x}_s, t)$ are unknown. Thus, the solution for any instantaneous $\mathbf{U}$ and drop position will depend on the prior history of the drop motion, as reflected by the interface and the drop shapes at the present time. Although this nonlinear interface and drop deformation problem cannot be solved exactly (except by numerical methods), it can be solved approximately by the method of "domain perturbations" when the deformations of both the drop and the interface are asymptotically small. The obvious physical requirement for this condition to be satisfied in the creeping-flow limit (i.e. $\mathrm{Re} \ll 1$) is that either

$$\mathrm{Ca}^* \text{ (or Cg}^*) \ll 1 \quad \text{and} \quad \mathrm{Ca} \text{ (or Cg)} \ll 1 \qquad [9]$$

or

$$\epsilon = \frac{1}{d} \ll 1. \qquad [10]$$

When either of the conditions [9] or [10] is satisfied, the interface and drop deformations will not only be in a quasi-steady state [i.e. $\eta(\mathbf{x}_s, t) = \eta(\mathbf{x}_s)$ and $f(\theta, \phi, t) = f(\theta, \phi)$], but the magnitudes of the deformations will also be asymptotically small.

In the present paper, we extend the singularity method of Lee *et al.* (1979) to consider the translation of a drop through a quiescent fluid for the asymptotic limit

$$\epsilon = \frac{1}{d} \ll 1.$$

As we shall see shortly in sections 3 and 4, the orders of magnitude of the interface deformation in this case are $O(\epsilon^2)$ and $O(\epsilon^3)$ for translations perpendicular and parallel to the interface, respectively, while that for drop deformation is $O(\epsilon^2)$ for an arbitrary translational motion. In the asymptotic limit, $\epsilon \ll 1$, the problem can be analyzed completely in terms of an asymptotic expansion for small ϵ, in which

$$f(\theta, \phi) = \epsilon^2 f^{(1)}(\theta, \phi) + \epsilon^3 f^{(2)}(\theta, \phi) + \cdots \quad [11]$$

and

$$\eta(\mathbf{x}_s) = \epsilon^m \eta^{(1)}(\mathbf{x}_s) + \epsilon^{m+1} \eta^{(2)}(\mathbf{x}_s) + \cdots. \quad [12]$$

The zeroth-order approximation [i.e. for $f(\theta, \phi) = \eta(\mathbf{x}_s) = 0$] thus represents the motion of a *spherical* drop near a *flat* interface. When the velocity and stress fields have been determined at the zeroth-order approximation, the normal stress conditions [6] and [8] can be used to determine a first nonzero approximation to the deformation of both the interface and the drop.

Due to the linearity of Stokes' equation and boundary conditions for the small deformation limit (i.e. $\epsilon \ll 1$), the solutions of two basic problems (i.e. translation normal or parallel to the interface) are sufficient to determine the drop and fluid motions for any arbitrary applied force on the drop. Further, in the zeroth-order approximation for $f(\theta, \phi) = \eta(\mathbf{x}_s) = 0$, the singularity method can be simplified to the superposition of fundamental solutions for a point force (i.e. Stokeslet), a potential dipole and higher order singularities (e.g. a stresslet, a rotlet, a potential quadrupole etc.) at the center of the drop (cf. Chwang & Wu 1975). Thus, solutions for the zeroth-order problem are constructed in the following manner. First, we put singularities at the center of the drop which satisfy exactly the boundary conditions at the drop surface in an unbounded fluid. The resulting unbounded-domain solution does *not* satisfy the boundary conditions at the flat interface; instead, an error of $O(\epsilon)$ is generated at the interface. To eliminate this "error", the simple transformation rule of Lee *et al.* (1979) is used to obtain corresponding fundamental singularity solutions that satisfy precisely the boundary conditions at the interface. In general, however, these new solutions do not satisfy boundary conditions any longer at the drop surface, but induce an error of $O(\epsilon)$. Additional higher-order singularities must then be included at the center of the drop to cancel the induced error of $O(\epsilon)$ at the drop surface, and so on. The result of this procedure is an asymptotic approximation, in the form of a series in ϵ, that is valid in the limit $\epsilon \to 0$. The asymptotic approximation solution, to the zeroth-order problem for translation of a *spherical* fluid drop either perpendicular or parallel to a *flat* interface, is then used to calculate the first corrections, $f^{(1)}(\theta, \phi)$ and $\eta^{(1)}(\mathbf{x}_s)$, for the shapes of the fluid drop and the plane interface for each case.

Let us now turn to the method of determining the deformation-induced lateral migration of a drop in translation parallel to an interface. An analytical approach based upon the reciprocal theorem of Lorentz (cf. Happel & Brenner 1965) provides the most efficient method because it allows the lateral migration velocity to be calculated from the zeroth-order solution, without any need to determine the first-order contribution to the velocity and pressure fields in each fluid. The analysis was discussed in considerable detail by Chan & Leal (1979) in connection with the problem of non-Newtonian and deformation-induced migration of a drop, and will thus be displayed here only in outline. According to the reciprocal theorem,

$$\int_{\mathbf{A}} d\mathbf{A} \cdot \boldsymbol{\sigma} \cdot \tilde{\mathbf{u}} = \int_{\mathbf{A}} d\mathbf{A} \cdot \tilde{\boldsymbol{\sigma}} \cdot \mathbf{u}, \quad [13]$$

where $(\mathbf{u}, \boldsymbol{\sigma})$ and $(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\sigma}})$ represent the velocity and stress fields corresponding to two creeping flows of the same fluid contained by the same bounding surface, $\mathbf{A}$. To apply the reciprocal theorem to the calculation of lateral migration velocity, we must first consider the "so-called" complementary problem of the motion of a drop translating perpendicularly to an interface in a quiescent fluid. In general, the use of the reciprocal theorem requires that the complementary Stokes-flow problem be solved for a drop and an interface of the same shapes as the "real" ones under consideration—which must themselves be calculated from the normal stress balances on the drop surface and the interface in the full disturbance-flow Stokes problem of drop motion. In order to simplify the application of boundary conditions at the interface and on the drop surface, it is advantageous to use the method of domain perturbations to approximate all quantities that are to be evaluated at the surfaces of the *deformed* interface and drop, in terms of equivalent quantities evaluated at $\eta(\mathbf{x}_s)=0$ and $f(\theta,\phi)=0$ using a Taylor series expansion about $\eta(\mathbf{x}_s)=0$ and $f(\theta,\phi)=0$, for $\epsilon \ll 1$. Hence, in effect, we replace the original problem, which has boundary conditions at the *deformed* interface and *deformed* drop surface, with an equivalent problem, for $\epsilon \ll 1$, in which modified boundary conditions are applied at the surfaces of a *flat* interface and a *spherical* drop. Although the complementary problem must normally be solved for a drop and an interface of the same shapes as the real ones under consideration, the reduction of the full problem to the motion of a *spherical* drop near a *flat* interface with a set of modified boundary conditions means that we can also use the solution for a spherical drop and a flat interface for the complementary problem.

An expression for the lateral migration velocity $\mathbf{U}_{\mathrm{lm}}$ can therefore be obtained from the reciprocal theorem, by first combining the reciprocal-theorem integrals for the two fluids, with the complementary Stokes flow identified as $(\tilde{\mathbf{u}},\tilde{\boldsymbol{\sigma}})$ and the disturbance flow generated by parallel translation of the drop identified as $(\mathbf{u},\boldsymbol{\sigma})$, and then applying the modified boundary conditions to evaluate the integrands of the combined integrals over S and H that result. The result in this case is

$$\begin{aligned}\mathbf{U}_{\mathrm{lm}}\cdot\int_S \tilde{\boldsymbol{\sigma}}\cdot\mathbf{n}^*\,\mathrm{d}A = &-\int_S\left[\!\left[\left\{-f\left[\left|\frac{\partial}{\partial r}\boldsymbol{\sigma}\right|\right]_S\cdot\mathbf{e}_r+\nabla f\cdot[|\boldsymbol{\sigma}|]_S+\frac{1}{\mathrm{Ca}^*}\nabla f(2f+\nabla^2 f)-\frac{1}{\mathrm{Cg}^*}f\nabla f\right\}\cdot\tilde{\mathbf{u}}\right.\right.\\ &\left.\left.+[|\tilde{\boldsymbol{\sigma}}|]_S:\mathbf{e}_r\mathbf{e}_r\left\{f\frac{\partial\mathbf{u}_2}{\partial r}\cdot\mathbf{e}_r-\mathbf{u}_2\cdot\nabla f\right\}+f\kappa\tilde{\boldsymbol{\sigma}}_3\cdot\mathbf{e}_r\cdot\left[\left|\frac{\partial\mathbf{u}}{\partial r}\right|\right]_S\right]\!\right]\mathrm{d}A\\ &-\int_H\left[\!\left[\left\{-\eta\left[\left|\frac{\partial}{\partial z}\boldsymbol{\sigma}\right|\right]_H\cdot\mathbf{e}_z+\nabla\eta\cdot[|\boldsymbol{\sigma}|]_H+\frac{1}{\mathrm{Ca}}\nabla^2\eta\,\nabla\eta-\frac{1}{\mathrm{Cg}}\eta\nabla\eta\right\}\cdot\tilde{\mathbf{u}}\right.\right.\\ &\left.\left.+[|\tilde{\boldsymbol{\sigma}}|]_H:\mathbf{e}_z\mathbf{e}_z\left\{\eta\frac{\partial\mathbf{u}_1}{\partial z}\cdot\mathbf{e}_z-\mathbf{u}_1\cdot\nabla\eta\right\}+\eta\cdot\tilde{\boldsymbol{\sigma}}_2\cdot\mathbf{e}_z\cdot\left[\left|\frac{\partial\mathbf{u}}{\partial z}\right|\right]_H\right]\!\right]\mathrm{d}A. \qquad [14]\end{aligned}$$

It can be noted that the effects of interface and drop deformation on drop motion are simply additive, as small corrections to the basic problem of the motion of a spherical drop near a flat interface, i.e. the results are thus the migration velocity for a deforming drop near a flat interface, plus the migration velocity for a spherical drop near a deforming interface. The expression [14] is somewhat complicted, but the lateral velocity at leading order is easily evaluated knowing only the first-order shape functions, $f^{(1)}(\theta,\phi)$ and $\eta^{(1)}(\mathbf{x}_s)$, the complementary flow field $(\tilde{\mathbf{u}},\tilde{\boldsymbol{\sigma}})$, plus the zeroth-order solution $(\mathbf{u},\boldsymbol{\sigma})$ for translation of a *spherical* drop parallel to a *flat* interface.

3. TRANSLATION PERPENDICULAR TO THE INTERFACE

3.1. Drag on a fluid drop

Let us begin by considering the translation of a *spherical* fluid droplet perpendicular to an infinite plane, *flat*, interface, i.e. the complementary problem for determining the deformation-induced migration of a drop. In an infinite fluid domain with no external boundaries, an exact solution for translation of a fluid droplet is the Hadamard–Rybczynski solution; (e.g. Happel & Brenner 1965). The velocity field *outside* the fluid drop in this solution can be represented as a superposition of the fundamental solutions for a point force (i.e. Stokeslet) and a potential dipole, both applied at the center of the drop. For a fluid droplet with viscosity $\kappa\mu_2$ (i.e. the viscosity ratio $\kappa = \mu_3/\mu_2$) which is moving with a constant velocity $\mathbf{U} = \mathbf{e}_z$, the required singularities are of the form:

$$\text{Stokeslet: } \tfrac{3}{4}A(\kappa)\,\mathbf{u}_S(\mathbf{x};\mathbf{e}_z)$$

and

$$\text{Potential dipole: } -\tfrac{1}{4}B(\kappa)\,\mathbf{u}_D(\mathbf{x};\mathbf{e}_z),$$

where $\mathbf{u}_S(\mathbf{x};\mathbf{e}_z)$ and $\mathbf{u}_D(\mathbf{x};\mathbf{e}_z)$ denote the fundamental solutions for a Stokeslet $\mathbf{e}_z$ and a potential dipole $\mathbf{e}_z$ located at the center of the drop in an unbounded single-fluid domain (cf. Chwang & Wu 1975). The parameters $A(\kappa)$ and $B(\kappa)$ are defined as

$$A(\kappa) = \frac{\frac{2}{3}+\kappa}{1+\kappa}, \qquad B(k) = \frac{\kappa}{1+\kappa} \qquad [15]$$

and clearly depend on the viscosity ratio κ. When either $\kappa \to 0$ or $\kappa \to \infty$, these parameters reduce to the values for an inviscid gas bubble or a rigid sphere, respectively.

Since we consider only the limit of $\epsilon \ll 1$, the solution of the full problem, including the interface, is most conveniently obtained via the method of reflections, as explained in some detail by Lee *et al.* (1979). The zeroth-order approximation in this procedure, $\mathbf{u}_2^{(0)}$, is the single-fluid unbounded domain solution:

$$\mathbf{u}_2^{(0)}(\mathbf{x}) = \tfrac{3}{4}A(\kappa)\mathbf{u}_S(\mathbf{x};\mathbf{e}_z) - \tfrac{1}{4}B(\kappa)\mathbf{u}_D(\mathbf{x};\mathbf{e}_z). \qquad [16]$$

Here, in the notation of $\mathbf{u}_2^{(j)}$, the superscript (j) indicates the level of approximation in the context of the method of reflections. Though the zeroth-order approximation [16] in the procedure exactly satisfies the boundary conditions at the drop surface (i.e. continuity of tangential velocity and stress plus the zero normal velocity), it does not satisfy boundary conditions at the flat interface. However, a first correction $\mathbf{u}_2^{(1)}$ which does satisfy these conditions can be obtained by simply utilizing the same form [16] as in the zeroth-order solution, but with the fundamental solutions $\mathbf{u}_S$ and $\mathbf{u}_D$ replaced by the corresponding fundamental solutions for a point force etc. in the presence of the flat interface, obtained by the generalized reciprocal theorem of Lee *et al.* (1979). It is convenient to express this solution in the form, $\mathbf{u}_2^{(0)} + \mathbf{u}_2^{(1)}$, as a sum of the zeroth-order solution plus a "correction". Although this combined solution satisfies the interface boundary conditions, it now does not satisfy the boundary conditions at the drop surface, and additional singularities are needed at the drop center in order to cancel the velocity field correction $\mathbf{u}_2^{(1)}$ at the drop surface: namely, the interface reflection of the Stokeslet and the potential dipole, both of which are nonzero on the drop surface. The preceding two steps, leading to the approximate solution, $\mathbf{u}_2^{(0)} + \mathbf{u}_2^{(1)}$, can be carried out for arbitrary ϵ. However, the resulting expression for $\mathbf{u}_2^{(1)}$ at the drop surface is highly complicated, and it is not possible for arbitrary ϵ to determine singularities at the drop center which precisely satisfy the continuity of tangential velocity and stress and zero normal velocity boundary conditions at all points on the drop surface. Instead, we consider the asymptotic limit $\epsilon \ll 1$, and then choose singularities to cancel only the first few terms of $\mathbf{u}_2^{(1)}$ at the drop surface, with $\mathbf{u}_2^{(1)}$ expressed in powers of ϵ. The leading terms of $\mathbf{u}_2^{(1)}$ near the drop surface, for small ϵ, are

$$\mathbf{u}_2^{(1)}(\mathbf{x}) = \tfrac{9}{8}A(\kappa)A(\lambda)[-\epsilon\cdot\mathbf{e}_z + \epsilon^2\cdot\tfrac{1}{4}\mathbf{E}\cdot\mathbf{x}] + O(\epsilon^3), \qquad [17]$$

in which λ is the viscosity ratio (i.e. $\lambda = \mu_1/\mu_2$) of the two continuous fluid phases 1 and 2. The strain rate tensor $\mathbf{E}$ has Cartesian components, $E_{ij} = \delta_{ij} - 3\delta_{i3}\delta_{j3}$, with the origin at the center of the drop.

Insofar as [17] is concerned, the presence of the interface is thus equivalent to an induced steady streaming flow at $O(\epsilon)$ in the direction opposite to that of the drop motion. The term of $O(\epsilon^2)$ in [17] is equivalent to an axisymmetric uniaxial extensional flow with a stagnation point at the drop center, and the z-axis as the symmetry axis. The singularities required to cancel the additional velocity field $\mathbf{u}_2^{(1)}(\mathbf{x})$ of [17] at the drop surface can be readily evaluated. We have seen previously that a uniform velocity at the drop surface can be generated by superposition of a Stokeslet and a potential dipole. It can be shown that an extensional flow of the type represented by the $O(\epsilon^2)$ term in [17] is generated in an unbounded single fluid by superposition of a stresslet and a potential quadrupole. To counter the terms of $O(\epsilon^2)$ in [17], we thus require the superposition of a stresslet and a potential quadrupole at the drop center. The resulting velocity field is

$$\mathbf{u}_{2,\mathrm{EX}}^{(2)} = \tfrac{1}{2}G_{\mathrm{EX}}[B(\kappa)\mathbf{u}_{\mathrm{PQ}}(\mathbf{x};\mathbf{e}_z,\mathbf{e}_z) + 5C(\kappa)\mathbf{u}_{\mathrm{SS}}(\mathbf{x};\mathbf{e}_z,\mathbf{e}_z)], \qquad [18]$$

where G_{EX} is the strain rate of the reflected extensional flow and is given by

$$G_{\mathrm{EX}} = \epsilon^2 \tfrac{9}{32}A(\kappa)A(\lambda) \qquad [19]$$

and the parameter $C(\kappa)$ is defined as

$$C(\kappa) = \frac{\frac{2}{5}+\kappa}{1+\kappa}. \qquad [20]$$

Here, $\mathbf{u}_{\mathrm{PQ}}$ and $\mathbf{u}_{\mathrm{SS}}$ denote the fundamental solutions for a potential quadrupole and a stresslet located at the drop center in an *unbounded* fluid. Thus, the complete contribution to the velocity field that is required to cancel the first two terms of $\mathbf{u}_2^{(1)}$ at the drop surface is a superposition of

$$\text{Stokeslet: } \tfrac{3}{4}A(\kappa)\mathbf{u}_{\mathrm{S}}(\mathbf{x};\mathbf{e}_z)\left[\sum_{n=0}^{2}\{\tfrac{9}{8}A(\kappa)A(\lambda)\epsilon\}^n + O(\epsilon^3)\right], \qquad [21]$$

$$\text{Potential dipole: } -\tfrac{1}{4}B(\kappa)\mathbf{u}_{\mathrm{D}}(\mathbf{x};\mathbf{e}_z)\left[\sum_{n=0}^{2}\{\tfrac{9}{8}A(\kappa)A(\lambda)\epsilon\}^n + O(\epsilon^3)\right], \qquad [22]$$

$$\text{Stresslet: } \epsilon^2 \tfrac{45}{64}A(\kappa)C(\kappa)A(\lambda)\mathbf{u}_{\mathrm{SS}}(\mathbf{x};\mathbf{e}_z,\mathbf{e}_z) \qquad [23]$$

and

$$\text{Potential quadrupole: } \epsilon^2 \tfrac{9}{64}A(\kappa)B(\kappa)A(\lambda)\mathbf{u}_{\mathrm{PQ}}(\mathbf{x};\mathbf{e}_z,\mathbf{e}_z). \qquad [24]$$

The complete velocity field, $\mathbf{u}_2^{(0)} + \mathbf{u}_2^{(1)} + \mathbf{u}_2^{(2)}$, with $\mathbf{u}_2^{(2)}$ resulting from the superposition of [21]–[24], now satisfies boundary conditions exactly at the interface and boundary conditions to $O(\epsilon^3)$ at the drop surface. Higher-order approximations could be obtained by a straightforward continuation of the same procedure. However, the solution above is sufficient for present purposes.

Bart (1968) obtained an exact result for the drag force on a small spherical drop settling toward a flat interface between two immiscible viscous fluids, using the eigensolutions of Laplace's equation in bipolar coordinates. It is a simple matter to calculate the approximate drag force $F_3\mathbf{e}_z$ on the fluid droplet from the present asymptotic solution. The drag ratio, i.e. the drag divided by the Stokes drag $6\pi\mu_2 aU$, is simply given as

$$\frac{F_3}{6\pi\mu_2 aU} = A(\kappa)\sum_{n=0}^{2}\{\tfrac{9}{8}A(\kappa)A(\lambda)\epsilon\}^n + O(\epsilon^3). \qquad [25]$$

In figure 2 the drag ratio [25] is plotted as a function of d, the distance between the drop and the interface, for the three values of $\lambda = 0$, 1 and ∞. In particular, we choose $\mu_1 = \mu_3$ (i.e. $\kappa = \lambda$), which is relevant to the final phase-separation stages of a liquid–liquid extraction process in which droplets of fluid 1 rise towards a stationary interface through a second fluid 2, and droplets of fluid 2 settle through fluid 1. The "exact" drag ratios numerically calculated by Bart (1968) are also shown in figure 2. There is very good agreement between the two solutions, except in the region near $d \approx 1$. As expected, the discrepancy between the two results becomes larger as the drop approaches the interface owing to the poor convergence of the asymptotic solution [25] in powers of ϵ. However, a detailed comparison shows that, even for $d \approx 2$, there is very good agreement between the two solutions, and the relative error in the asymptotic solution [25], compared to the exact solution of Bart (1968), is $<5\%$ for $d \geqslant 2.5$. It should be noted from [25] that the interface effect on the drop becomes stronger with an increase in λ (or κ) since $A(\lambda)$ [or $A(\kappa)$] is a monotonically increasing function of the viscosity ratio λ (or κ).

3.2. *Drop and interface deformations*

When a fluid drop moves near a fluid interface, the fluid in the neighborhood of the drop is disturbed. The disturbance generates a stress system which can be resolved into tangential and normal stresses acting at the plane interface and the drop surface. The tangential stresses are assumed to be transmitted undiminished across the interfaces and thus establish a system of velocity gradients in the vicinity of the interfaces. The normal stresses, on the other hand, are discontinuous at the plane interface and the drop surface, and generate normal stress differences across the corresponding interfaces that can only be balanced by capillary or body forces through interface deformation. The zeroth-order solutions obtained in the preceding section 3.1 for a *spherical* drop [i.e. $f(\theta, \phi) = 0$] translating near a *flat* interface [i.e. $\eta(\mathbf{x}_s) = 0$] satisfy the conditions of continuity of the tangential velocity and stress at the undeformed interfaces, as well as zero normal velocity. However, they do produce an imbalance in the normal stress components across the plane interface and the spherical drop surface. Thus, to calculate a first correction to the interface and the drop shapes, it is necessary to solve the differential equations [6] and [8] with the normal stress differences $[|\boldsymbol{\sigma} \cdot \mathbf{n}|]$ evaluated using the zeroth-order solution.

The normal stress difference $[|\boldsymbol{\sigma} \cdot \mathbf{n}^*|]_S$ across the drop surface S can be evaluated from the zeroth-order solution and expressed in terms of the Legendre polynomial of the second order, P_2:

$$[|\boldsymbol{\sigma} \cdot \mathbf{n}^* \cdot \mathbf{n}^*|]_S = p^{\text{in}} - p^{\infty} + 16 G_{\text{EX}} \frac{16 + 19\kappa}{16 + 16\kappa} P_2(\cos\theta), \qquad [26]$$

where θ is the spherical polar angle measured from the z-axis, i.e. the axis of symmetry in the induced straining flow with strain rate G_{EX} given by [19]. In [26], p^{in} denotes the pressure at the interface inside the drop phase and p^{∞} is the reference pressure far from the drop. This pressure difference $p^{\text{in}} - p^{\infty}$, is precisely balanced by interfacial tension for the drop in its undeformed spherical shape ($S: r - 1 = 0$), i.e.

$$p^{\text{in}} - p^{\infty} = \frac{2}{\text{Ca}^*}. \qquad [27]$$

It is thus obvious, from [19], [26] and [27], that the first correction to the drop shape, $f(\theta, \phi)$, is $O(\epsilon^2)$ and the independent of the azimuthal angle ϕ owing to the axisymmetry of the problem. The differential equation for the shape function $f(\theta)$ at $O(\epsilon^2)$ in the asymptotic form of [11] follows directly by substitution of [26] and [27] into [6], noting that $\mathbf{n}^* = \nabla S / |\nabla S|$, so that

$$\frac{1}{\sin\theta} \frac{1}{\partial\theta} \left[\sin\theta \frac{\partial}{\partial\theta} f^{(1)} \right] + 2 f^{(1)} = -\text{Ca}^* A(\kappa) A(\lambda) \frac{9}{2} \cdot \frac{16 + 19\kappa}{16 + 16\kappa} P_2(\cos\theta). \qquad [28]$$

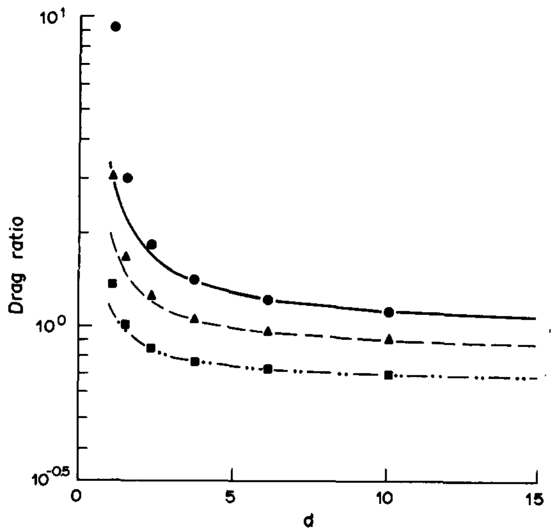

Figure 2. Drag ratio as a function of the distance, d, between the drop center and the interface for normal translation: ——, for $\lambda = \kappa \to \infty$; ----, for $\lambda = \kappa = 1$; —··—, for $\lambda = \kappa = 0$; markers are the corresponding exact-solution results of Bart (1968).

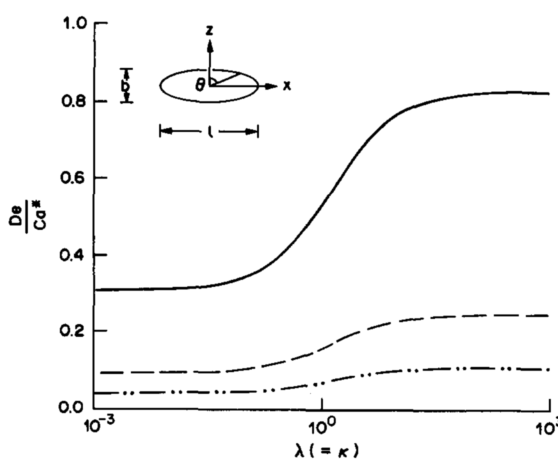

Figure 3. Variation of drop deformation, De/Ca*, as a function of viscosity ratios $\lambda (= \kappa)$ for normal translation: ——, for $d = 1.1$; ----, for $d = 2$; —··—, for $d = 3$.

Equation [28] can be solved in terms of the Legendre polynomial P_2, subject to the conditions

$$\int_{-1}^{1} f^{(1)}(\theta)\,\mathrm{d}(\cos\theta) = 0 \quad \text{for } \max|\epsilon^2 f^{(1)}(\theta)| \ll 1,$$

since the characteristic length a has been set equal to the radius of the "equivalent" spherical drop, and

$$\int_{-1}^{1} \cos\theta \cdot f^{(1)}(\theta)\,\mathrm{d}(\cos\theta) = 0 \quad \text{for } \max|\epsilon^2 f^{(1)}(\theta)| \ll 1,$$

since the origin of the coordinate system has been chosen to coincide with the center of mass of the drop. The resulting solution for the drop shape $S(r, \theta)$ is

$$S(r, \theta)\colon r - 1 - f(\theta) = 0, \qquad [29]$$

where

$$f(\theta) \approx \epsilon^2 f^{(1)}(\theta) = -\epsilon^2 \mathrm{Ca}^* A(\kappa)A(\lambda)\frac{9}{8}\cdot\frac{16+19\kappa}{16+16\kappa}P_2(\cos\theta). \qquad [30]$$

The shape of a fluid drop translating perpendicularly to a plane interface is thus an oblate spheroid with the axis of revolution parallel to the direction of motion. It should be noted that the value of

$$A(\kappa)A(\lambda)\frac{9}{8}\cdot\frac{16+19\kappa}{16+16\kappa}$$

is a monotonically increasing function of λ and κ, and varies from 0.5 to 1.336 for all ranges of λ and $\kappa(\geqslant 0)$. Further, the first correction to the drop shape, $f(\theta)$, is of $O(\epsilon^2\mathrm{Ca}^*)$.

A convenient method for expressing the result for drop deformation is to measure l, the length of the drop in the direction of the major axis, and b, the breadth in the direction of the minor axis of the ellipsoid, see figure 3. Following Taylor (1934), it is useful to define a dimensionless *deformation* parameter, De, as

$$\mathrm{De} = \frac{l-b}{l+b}. \qquad [31]$$

This parameter De vanishes for a spherical drop, and is asymptotically unity for a long slender drop. The parameter De, in this case, can be easily evaluated from [19], [30] and [31]:

$$\mathrm{De} = \frac{l-b}{l+b} = \frac{3\mathrm{Ca}^* G_{\mathrm{EX}}\dfrac{16+19\kappa}{16+16\kappa}}{1-\mathrm{Ca}^* G_{\mathrm{EX}}\dfrac{16+19\kappa}{16+16\kappa}}$$

$$\approx 3\mathrm{Ca}^* G_{\mathrm{EX}}\frac{16+19\kappa}{16+16\kappa} \quad \text{as } \mathrm{Ca}^*\epsilon^2 \to 0. \qquad [32]$$

In figure 3, the dimensionless parameter De is illustrated as a function of the viscosity ratios λ and κ, with the condition $\lambda = \kappa$, for three values of the separation distance, $d = 1.1$, 2 and 3. As expected, for a given value of the capillary number, Ca*, the magnitude of the drop deformation (De), due solely to the presence of an interface (i.e. De $\to 0$ as $d \to \infty$), is increased with an increase in λ (or κ), and this effect is a strong function of the drop position relative to the interface.

Let us now turn to the *interface* shape, which can be obtained using the normal stress jump condition [8]. In order to proceed analytically, we assume that the deformation, $\eta(\mathbf{x}_s)$, is small, and that it can be represented asymptotically in the form [12], where ϵ is the small parameter of the problem, as defined in [10]. It is convenient for formulation purposes to utilize a cylindrical coordinate system (ρ, ϕ, z), with $z = d$ corresponding to the plane of the undeformed interface and the z-axis passing through the center of the drop at the origin.

The normal stress imbalance across the interface H at $z = d$, for translation of the drop normal to the interface, can be evaluated from the zeroth-order velocity and pressure fields, i.e.

$$[|\boldsymbol{\sigma} \cdot \mathbf{n} \cdot \mathbf{n}|]_H = \epsilon^2 \frac{9A(\kappa)}{\left(\frac{\rho^2}{d^2} + 1\right)^{5/2}} + O(\epsilon^3). \qquad [33]$$

It can easily be seen, from [33], that the first correction of the interface shape is $O(\epsilon^2)$, i.e. $m = 2$ in the asymptotic expansion of [12]. We now proceed to solve [8] for the shape of the interface, $\eta(\mathbf{x}_s)$ at $O(\epsilon^2)$, i.e. $\eta^{(1)}(\mathbf{x}_s)$. After combining [33] with [8] and [12], we obtain the following differential equation in cylindrical coordinates for the correction function $\eta^{(1)}(\mathbf{x}_s)$:

$$\nabla^2\eta^{(1)}(\mathbf{x}_s) - \Phi\eta^{(1)}(\mathbf{x}_s) = -\frac{9A(\kappa)\,\mathrm{Ca}}{\left(\frac{\rho^2}{d^2} + 1\right)^{5/2}}, \qquad [34]$$

where Φ is the Bond number.

The solution of [34] is straightforward and may be obtained either by use of the Green's function or by the variation of parameters technique. The resulting solution for the shape of the interface H is

$$H: z - d - \eta(\mathbf{x}_s) = 0, \qquad [35]$$

where

$$\eta(\mathbf{x}_s) \approx \epsilon^2\eta^{(1)}(\mathbf{x}_s)$$
$$= 9\epsilon^2\mathrm{Ca}\,A(\kappa)\left[I_0(\Phi^{1/2}\rho)\int_\rho^\infty \frac{\zeta K_0(\Phi^{1/2}\zeta)}{\left(\frac{\zeta^2}{d^2}+1\right)^{5/2}}\,\mathrm{d}\zeta + K_0(\Phi^{1/2}\rho)\int_0^\rho \frac{\zeta I_0(\Phi^{1/2}\zeta)}{\left(\frac{\zeta^2}{d^2}+1\right)^{5/2}}\,\mathrm{d}\zeta\right], \qquad [36]$$

in which I_0 and K_0 are the modified Bessel functions of order 0 of the first and second kinds, respectively. Equation [36] indicates that, up to the first correction of the interface shape at $O(\epsilon^2)$, the interface deformation is *independent* of the viscosity ratio λ of the two continuous bulk fluids, and only a weakly increasing function of the viscosity ratio κ between the drop and the continuous fluid phases. Specifically, $A(\kappa)$ varies from $\frac{2}{3}$ to 1 for the entire range of $\kappa \geqslant 0$. It is also noteworthy that, since the term in square brackets in [36] is $O(1)$ for *nonzero* Bond numbers, i.e.

$$\Phi = \frac{\mathrm{Ca}}{\mathrm{Cg}} = \frac{ga^2(\rho_2 - \rho_1)}{\gamma_{12}} > 0, \qquad [37]$$

the magnitude of the drop deformation is $O(\epsilon^2\mathrm{Ca})$.

Let us analyze the solution [36] in some detail by considering the relative effectiveness of the interfacial tension forces and the gravity forces in restricting the degree of interface deformation. In the differential equation [34] for the interface shape function, $\eta(\mathbf{x}_s)$ at $O(\epsilon^2)$, gravity acts directly on the degree of displacement from the undeformed shape, while the effect of interfacial tension is an indirect consequence of limiting the curvature of the interface. The obvious parameter for investigation of this question is Φ, a measure of the relative magnitude of the two forces. Figure 4 shows the maximum displacement, $\eta_{\max}$, which occurs at $\rho = 0$, as a function of Φ, for a viscosity ratio $\kappa = 1$, and a dimensionless distance $d = 3$ between the drop center and the plane ($z = d$) of the undeformed interface. In this plot, we carried out calculations for three fixed values of the capillary number, Ca = 0.01, 1 and 100, in order to make comparisons between the results for different values of Φ as meaningful as possible. It can be seen, from figure 4, that the magnitude of the interface distortion is remarkably sensitive to Φ, particularly in the limit of $\Phi \to 0$ (i.e. $\rho_1 \to \rho_2$). This is a consequence of the fact that the gravity force due to the density difference is much more effective than the surface tension in restoring the interface to a flat configuration, as previously indicated by Lee & Leal (1982). In figure 5, the maximum displacement, $\eta_{\max}$, is plotted as a function of the separation distance, d, for a capillary number Ca = 1, a viscosity ratio $\kappa = 1$, and three values of the Bond number $\Phi = 0.01$, 1 and 100. It is obvious from figure 5 that the degree of interface deformation rapidly increases as either the drop approaches the interface or as Φ decreases. The limiting case of $\Phi \to 0$ corresponds to surface-tension-dominated deformation.

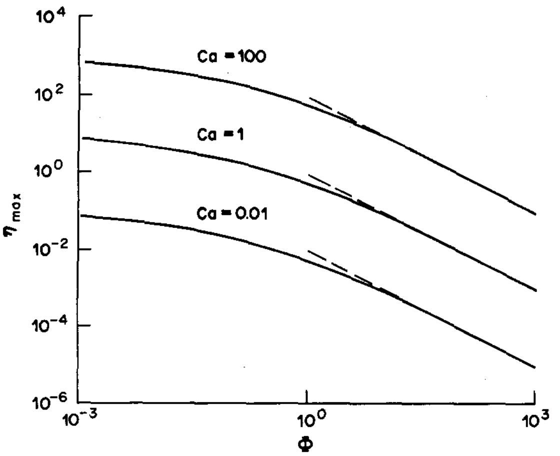

Figure 4. Maximum displacement, η_{max}, as a function of the Bond number, Φ, for normal translation: ——, for the exact numerical integration of [36]; ----, for the asymptotic results of [39].

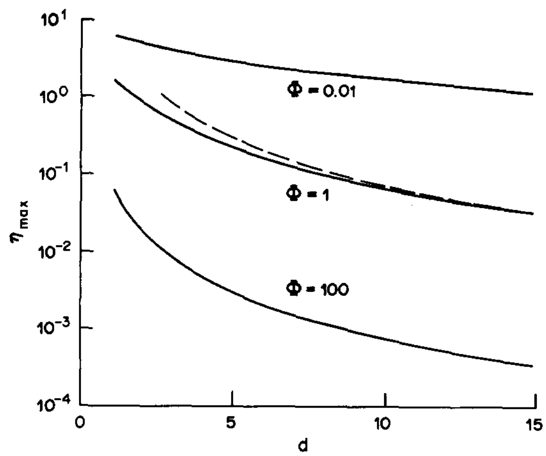

Figure 5. Maximum displacement, η_{max}, as a function of the separation distance, d, between the drop center and the interface for normal translation: ——, for the exact numerical integration of [36]; ----, for the asymptotic results of [39].

In the earlier studies of interface distortion via spontaneous fluctuations, Buff *et al.* (1965), Jhon *et al.* (1978), Zielinska & Bedeaux (1982) and Yang (1985) have noted that the magnitude of interface fluctuations diverges *logarithmically* in the limit $\Delta\rho \to 0$ with $\gamma_{12} = O(1)$, i.e. $\Phi \to 0$ with $\mathrm{Ca} = 0(1)$. This *weak* divergence is related to the fact that, in the classical hydrodynamic result, finite stable capillary waves must always exist if a nonuniform density distribution exists. In this limit, $\Phi \to 0$, the asymptotic behavior of the solution [36] is simply given as

$$\eta(\mathbf{x}_s) = 3\epsilon^2 \mathrm{Ca}\, A(\kappa) \ln \left[\frac{\dfrac{\epsilon}{\Phi^{1/2}}}{1 + \left(\dfrac{\rho^2}{d^2} + 1\right)^{1/2}} \right]$$

$$= O(\mathrm{Ca} \ln \Phi) \quad \text{as } \Phi \to 0, \qquad [38]$$

for fixed ϵ. Thus, the interface displacement has a log singularity if $\Phi \to 0$ with $\mathrm{Ca} = O(1)$, i.e. $\Delta\rho \to 0$ with $\gamma_{12} = O(1)$. It is evident, however, that $\eta(\mathbf{x}_s)$ is *bounded* and $O(\mathrm{Ca} \ln \mathrm{Ca})$ even when $\Phi = 0$ with $\mathrm{Cg} = O(1)$, i.e. $\gamma_{12} \to \infty$ with $\Delta\rho = O(1)$. The boundedness of $\eta(\mathbf{x}_s)$ for this case is also true of thermal fluctuations of a fluid–fluid interface, as noted in Yang (1985).

In the gravity-dominated limit, i.e. $\Phi \to \infty$, the shape function is given by

$$\eta(\mathbf{x}_s) \approx \epsilon^2 \frac{9A(\kappa)\mathrm{Ca}}{\left(\dfrac{\rho^2}{d^2} + 1\right)^{5/2} \Phi} \quad \text{as } \Phi \to \infty. \qquad [39]$$

Thus, the magnitude of interface distortion is independent of the interfacial tension γ_{12} and depends only on the density difference. In figures 4 and 5, the asymptotic approximation [39] for the interface deformation is shown for comparison with the exact numerical integration of [36]. It can be seen that the asymptotic form provides an excellent approximation, even for $\Phi \approx 1$, and the two results are almost exactly coincident for $\Phi \geqslant 100$.

4. TRANSLATION PARALLEL TO THE INTERFACE

4.1. Drag on a fluid drop

We now turn to the case of a fluid drop translating with velocity $\mathbf{U} = \mathbf{e}_x$ parallel to an infinite plane fluid interface, which is located at $z = d$. The solution in an unbounded fluid, i.e. the

Hadamard–Rybczynski solution, is simply the superposition of a Stokeslet and a potential dipole both oriented in the direction of motion, as noted in the previous problem, i.e.

$$\mathbf{u}_2^{(0)}(\mathbf{x}) = \tfrac{3}{4}A(\kappa)\mathbf{u}_S(\mathbf{x}; \mathbf{e}_x) - \tfrac{1}{4}B(\kappa)\mathbf{u}_D(\mathbf{x}; \mathbf{e}_x), \tag{40}$$

in which $A(\kappa)$ and $B(\kappa)$ are defined in [15].

As is the preceding example, the first correction, $\mathbf{u}_2^{(0)} + \mathbf{u}_2^{(1)}$, for the presence of the interface does not satisfy the boundary condition at the drop surface, because the interface reflection $\mathbf{u}_2^{(1)}(\mathbf{x})$ is nonzero at the drop surface. Following section 3.1, we examine the leading terms of the reflected velocity field at the drop surface as a power series in ϵ:

$$\mathbf{u}_2^{(1)}(\mathbf{x}) = \epsilon \cdot \tfrac{9}{16}A(\kappa)D(\lambda)\mathbf{e}_x + \boldsymbol{\Gamma}(\lambda, \kappa) \cdot \mathbf{x}, \tag{41}$$

where the parameter $D(\lambda)$ is defined as

$$D(\lambda) = \frac{\tfrac{2}{3} - \lambda}{1 + \lambda}, \tag{42}$$

and the second-order shear-rate tensor $\boldsymbol{\Gamma}(\lambda, \kappa)$ is given by

$$\boldsymbol{\Gamma}(\lambda, \kappa) = \Gamma_{ij} = \epsilon^2 \tfrac{9}{32}A(\kappa)\begin{bmatrix} 0 & 0 & D(\lambda) \\ 0 & 0 & 0 \\ -A(\lambda) & 0 & 0 \end{bmatrix}. \tag{43}$$

It can be seen from [41] and [43] that the presence of the interface in this case is equivalent in its effect on the motion of the drop to a steady streaming flow at $O(\epsilon)$ parallel to the interface, and a linear shear flow at $O(\epsilon^2)$ either normal or parallel to the interface.

In order to satisfy the conditions of continuity of velocity and tangential stress and zero normal velocity at the drop surface, we need additional singularities at the drop center that produce a velocity field at the drop surface of opposite sign. For the term of $O(\epsilon)$, a Stokeslet and a potential dipole are required. The singularities required to counter the $O(\epsilon^2)$ contribution can be evaluated by determining the corresponding solution for the linear shear flow in an *unbounded* fluid domain. It can be shown that a stresslet and a potential quadrupole are necessary to produce such flows in an unbounded single-fluid domain. Thus,

$$\begin{aligned}\mathbf{u}_{2,\mathrm{SH}}^{(2)} = &-\Gamma_{13}[\tfrac{1}{6}B(\kappa)\mathbf{u}_{PQ}(\mathbf{x}; \mathbf{e}_x, \mathbf{e}_z) + \tfrac{5}{6}C(\kappa)\mathbf{u}_{SS}(\mathbf{x}; \mathbf{e}_x, \mathbf{e}_z)] \\ &-\Gamma_{31}[\tfrac{1}{6}B(\kappa)\mathbf{u}_{PQ}(\mathbf{x}; \mathbf{e}_z, \mathbf{e}_x) + \tfrac{5}{6}C(\kappa)\mathbf{u}_{SS}(\mathbf{x}; \mathbf{e}_z, \mathbf{e}_x)],\end{aligned} \tag{44}$$

in which the reflected shear components Γ_{13} and Γ_{31} are defined in [43].

Consequently, for the translation of a fluid drop with viscosity $\mu_3(=\kappa\mu_2)$ parallel to the interface, the singularities required at the center of the drop through $O(\epsilon^2)$ are:

$$\text{Stokeslet: } \tfrac{3}{4}A(\kappa)\mathbf{u}_S(\mathbf{x}; \mathbf{e}_x)\left[\sum_{n=0}^{2}\{-\tfrac{9}{16}A(\kappa)D(\lambda)\epsilon\}^n + O(\epsilon^3)\right], \tag{45}$$

$$\text{Potential dipole: } -\tfrac{1}{4}B(\kappa)\mathbf{u}_D(\mathbf{x}; \mathbf{e}_x)\left[\sum_{n=0}^{2}\{-\tfrac{9}{16}A(\kappa)D(\lambda)\epsilon\}^n + O(\epsilon^3)\right], \tag{46}$$

$$\text{Stresslet: } -\tfrac{5}{6}(\Gamma_{13} + \Gamma_{31})C(\kappa)\mathbf{u}_{SS}(\mathbf{x}; \mathbf{e}_x, \mathbf{e}_z) \tag{47}$$

and

$$\text{Potential quadrupole: } -\tfrac{1}{6}B(\kappa)[\Gamma_{13}\mathbf{u}_{PQ}(\mathbf{x}; \mathbf{e}_x, \mathbf{e}_z) + \Gamma_{31}\mathbf{u}_{PQ}(\mathbf{x}; \mathbf{e}_z, \mathbf{e}_x)]. \tag{48}$$

From this solution, we can easily determine the hydrodynamic force $F_1\mathbf{e}_x$ exerted on a fluid droplet located at an arbitrary point near an interface. The drag ratio (the drag divided by the Stokes drag $6\pi\mu_2 aU$) is simply given as

$$\frac{F_1}{6\pi\mu_2 aU} = A(\kappa)\sum_{n=0}^{2}\{-\tfrac{9}{16}A(\kappa)D(\lambda)\epsilon\}^n + O(\epsilon^3). \tag{49}$$

When $\kappa \to \infty$, [49] reduces to the drag ratio for the case of a solid sphere, and is identical with the results of Lee *et al.* (1979) and Yang & Leal (1984) to $O(\epsilon^2)$. It may be seen from [49] that there exists a critical viscosity ratio λ equal to 2/3, above which the drag force on the fluid droplet in the presence of an interface is larger than that in an unbounded infinite fluid. For $\lambda < 2/3$, the drag is less than it would be in an infinite fluid. The critical viscosity ratio ($\lambda = 2/3$) is *independent* of the viscosity ratio κ and the drop position relative to the interface. In many respects, the results are similar to those for parallel translation of a solid sphere obtained by Lee *et al.* (1979) and Yang & Leal (1984). O'Neill (1964) calculated the drag ratio for the motion of a *solid* sphere parallel to a plane solid wall (i.e. $\lambda = \kappa \to \infty$) using an eigenfunction expansion in bipolar coordinates. The result of [49] is plotted in figure 6 for the same set of parameters as in figure 2. Also shown for comparison are the corresponding drag ratios determined by O'Neill (1964). As mentioned previously, we presume $\epsilon \ll 1$ in the derivation of [49]. Thus, for $\epsilon \ll 1$ (i.e. $d \gg 1$), the asymptotic solution [49] coincides almost exactly with O'Neill's result, which is the exact solution for the translation of a rigid sphere parallel to a solid wall. Even for $d \approx 1.5$, the approximate solution shows reasonably good agreement with the exact solution. Indeed, the relative error is within 3.9% for $d \geqslant 1.5$.

The interface-induced rotation of the drop can be evaluated from the magnitude and orientation of the reflected shear flows. The angular velocity of the spherical fluid drop is equal to

$$\mathbf{\Omega} = \frac{1}{2}(\Gamma_{13} - \Gamma_{31})\mathbf{e}_y = \epsilon^2 A(\kappa)\frac{3}{16}\cdot\frac{1}{1+\lambda}\mathbf{e}_y + O(\epsilon^3). \quad [50]$$

This rotation can be viewed as a consequence of two different mechanisms: one is the dynamic effect due essentially to the difference in the shear stress above and below the drop; and the other is due primarily to the kinematic condition (i.e. zero normal velocity) at the interface. It will be noted that, when $\lambda \to \infty$, there is no term of $O(\epsilon^2)$ in $\mathbf{\Omega}$ for the entire range of $\kappa(\geqslant 0)$.

4.2. *Drop and interface deformations*

We begin with the shape deformation of the drop in translation parallel to the interface. It is well-known that the shape of a fluid droplet translating in an unbounded single-fluid domain is *spherical*, and thus the shape deformation is due solely to the presence of the interface. An analytical approach similar to that outlined in section 3 provides the most efficient method of determining the drop deformation. However, the deformed drop shape in translation parallel to the interface is not axisymmetric, as in the normal translation, owing to the asymmetry of the problem.

The velocity and stress in the neighborhood of the surface of a nearly spherical drop can be evaluated approximately by utilizing the corresponding solution for a *spherical* drop in translation

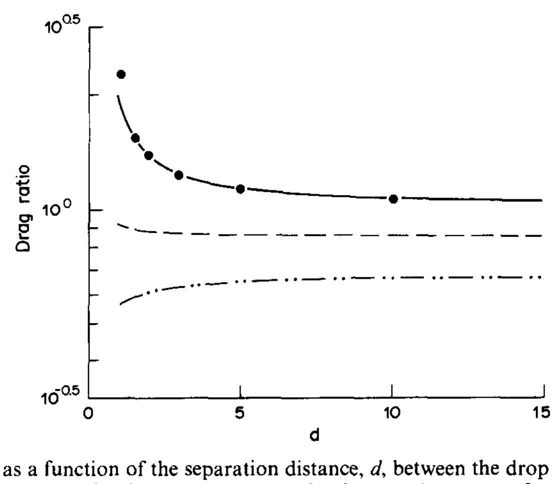

Figure 6. Drag ratio as a function of the separation distance, d, between the drop center and the interface for parallel translation: ——, for $\lambda = \kappa \to \infty$; – – – –, for $\lambda = \kappa = 1$; — · · —, for $\lambda = \kappa = 0$; markers are the corresponding exact-solution results of O'Neill (1964).

parallel to a *flat* interface. The normal stress difference across the drop surface S in the *equatorial plane* for a slightly deformed drop is

$$[|\sigma \cdot \mathbf{n}^* \cdot \mathbf{n}^*|]_S = p^{in} - p^{\infty} + 4(\Gamma_{13} + \Gamma_{31}) \frac{16 + 19\kappa}{16 + 16\kappa} \cos 2\phi_m, \quad [51]$$

where ϕ_m is the polar angle in the two-dimensional equatorial plane, cf. figure 7. Since the pressure difference $p^{in} - p^{\infty}$ of $O(1)$ is precisely balanced by the surface tension of the undeformed (spherical) drop (i.e. $S: r - 1 = 0$), the drop deformation, which is $O(\epsilon^2)$, results from the reflected linear shear flow with shear rate Γ_{13} or Γ_{31} either normal or parallel to the interface. It should be noted that the principal axes for the rate of strain (i.e. principal axes of distortion) generated by the stresslet and the potential quadrupole in [44] lag by the angle $\pi/4$ behind the x, z-axes. The normal stress difference has a maximum positive value when $\phi_m = 0$ and a maximum negative value when $\phi_m = \pi/2$, corresponding to the principal axes of distortion shown in figure 7 [note that $\Gamma_{13} + \Gamma_{31} \leqslant 0$ and is $O(\epsilon^2)$]. When $\Gamma_{13} + \Gamma_{31}$ is nonzero, the drop is deformed in such a way that the stress generated by the stresslet and the potential quadrupole is balanced by the interfacial tension, i.e. the curvature of the drop changes so as to satisfy the condition [6] for equilibrium. It can be shown that for small deformations this condition is satisfied when the *equator* of the drop surface S assumes an ellipsoidal form given by the polar equation

$$S: r - 1 - f(\phi_m) = 0, \quad [52]$$

where

$$f(\phi_m) = -\mathrm{Ca}^*(\Gamma_{13} + \Gamma_{31}) \frac{16 + 19\kappa}{16 + 16\kappa} \cos 2\phi_m. \quad [53]$$

The dimensionless deformation parameter, De, in this case, can be easily obtained from [43], [52] and [53], and is given by

$$\mathrm{De} = \frac{l - b}{l + b} = \epsilon^2 \mathrm{Ca}^* A(\kappa) B(\lambda) \frac{9}{16} \cdot \frac{16 + 19\kappa}{16 + 16\kappa}, \quad [54]$$

in which the functions A and B are defined in [15]. It can easily be seen that the magnitude of drop deformation is of $O(\epsilon^2 \mathrm{Ca}^*)$ and the drop remains nearly spherical in the present asymptotic limit of $\epsilon \ll 1$ with $\mathrm{Ca}^* = O(1)$. The asymptotic limit ensures that the principal strain rate of the interface reflections is sufficiently small that ultimately the distortion is limited by the relatively strong surface tension and has *inter alia* a very weak dependence on the viscosity ratio κ. In this weak

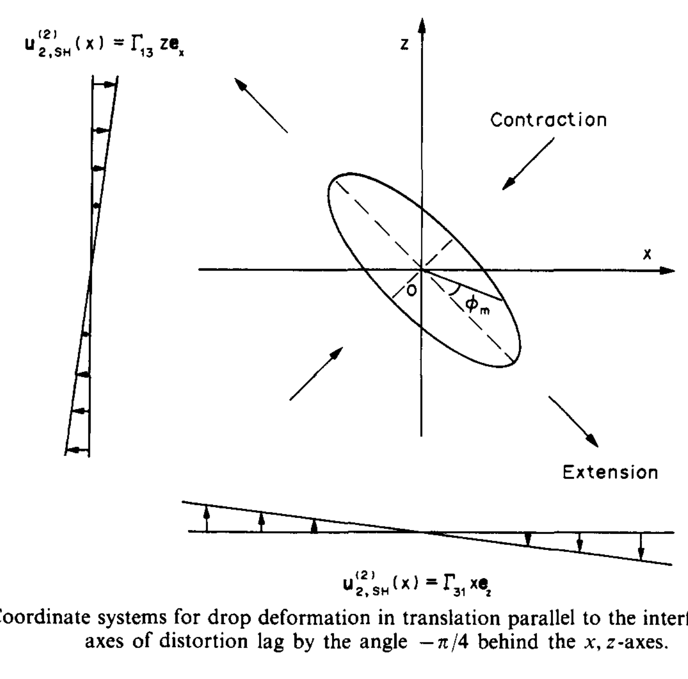

Figure 7. Coordinate systems for drop deformation in translation parallel to the interface. The principal axes of distortion lag by the angle $-\pi/4$ behind the x, z-axes.

flow limit, the interface-induced rotation is small enough, as noted in [50], to allow the drop sufficient time to accommodate its shape to the changing stress. Under these circumstances the principal axis of the drop is aligned with the direction of the principal rate of strain, i.e. the angle between the principal axis of the drop and the positive x-direction is $-\pi/4$, as illustrated in figure 7. It is known that the lateral position of a *spherical* drop in a Newtonian fluid at zero Reynolds number is *fixed* for all times by its value at some initial instant. As we shall see shortly in section 4.3, however, the introduction of reflected-flow-induced asymmetry in the drop shape and orientation relative to the interface in a bounded flow domain may lead to lateral migration of the drop in a direction normal to the interface, even for Newtonian fluids at zero Reynolds number.

In figure 8 the dimensionless deformation, De, is plotted as a function of the viscosity ratio λ with the condition of $\lambda = \kappa$ for three values of the separation distance $d = 1.1$, 2 and 3. The qualitative dependence of drop deformation on the viscosity ratio is similar to the case of normal translation to an interface. It may be noted that, when $\lambda \to 0$, there is no term of $O(\epsilon^2)$ in De. Indeed, the distortion effects produced by the reflected shear components, Γ_{13} and Γ_{31}, are exactly balanced up to $O(\epsilon^2)$ and the drop in translation parallel to a free boundary (i.e. $\lambda \to 0$) remains *spherical*, irrespective of the viscosity ratio κ between the drop and the continuous fluid phases.

Let us then consider the interface shape for translation of a fluid drop parallel to the plane, $z = d$. Since the velocity and pressure fields have been determined from the zeroth-order solution to the *complementary* problem for translation of a *spherical* drop in the presence of a *flat* interface, the normal stress condition [8] can again be used to determine a first approximation to the deviation of the interface shape from flat. In the asymptotic limit of $\epsilon \ll 1$, we obtain a normal stress difference for a fluid drop in parallel translation as follows:

$$[|\boldsymbol{\sigma} \cdot \mathbf{n} \cdot \mathbf{n}|]_H = \epsilon^3 \frac{9A(\kappa)x}{\left(\dfrac{\rho^2}{d^2} + 1\right)^{5/2}} + O(\epsilon^3). \qquad [55]$$

It can be noted that the first correction to the interface shape in this case is $O(\epsilon^3)$, i.e. $m = 3$ in the asymptotic expansion of [12]. After combining [55] with [8] and [12], we have

$$\nabla^2\eta^{(1)}(\mathbf{x}_s) - \Phi\eta^{(1)}(\mathbf{x}_s) = -\frac{9A(\kappa)\,\mathrm{Ca}\,x}{\left(\dfrac{\rho^2}{d^2} + 1\right)^{5/2}} \qquad [56]$$

for the first-order correction function $\eta^{(1)}(\mathbf{x}_s)$. This partial differential equation, expressed in cylindrical polar coordinates (ρ, ϕ, z) with $x = \rho\cos\phi$, can be solved via separation of variables and variation of parameter techniques. The resulting solution for the shape of the interface H is

$$H: z - d - \eta(\mathbf{x}_s) = 0, \qquad [57]$$

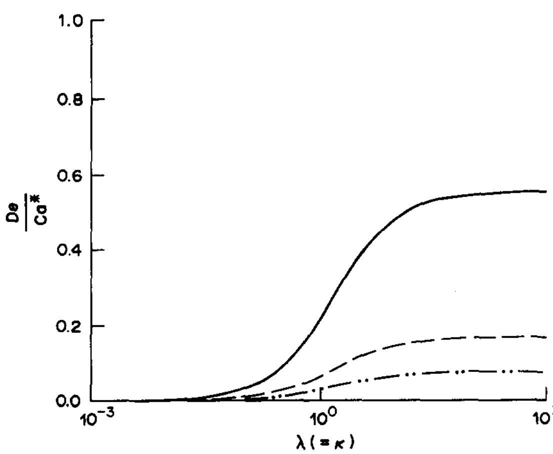

Figure 8. Variation of drop deformation, De/Ca*, as a function of viscosity ratios $\lambda(=\kappa)$ for parallel translation: ——, for $d = 1.1$; ----, for $d = 2$; —··—, for $d = 3$.

where

$$\eta(\mathbf{x}_s) \approx \epsilon^3 \eta^{(1)}(\mathbf{x}_s)$$

$$= \epsilon^3 \frac{9A(\kappa)\,\mathrm{Ca}\,x}{\rho}\left[I_1(\Phi^{1/2}\rho)\int_\rho^\infty \frac{\zeta^2 K_1(\Phi^{1/2}\zeta)}{\left(\frac{\zeta^2}{d^2}+1\right)^{5/2}}\mathrm{d}\zeta + K_1(\Phi^{1/2}\rho)\int_0^\rho \frac{\zeta^2 I_1(\Phi^{1/2}\zeta)}{\left(\frac{\zeta^2}{d^2}+1\right)^{5/2}}\mathrm{d}\zeta\right] \quad [58]$$

where I_1 and K_1 are the modified Bessel functions of order 1 of the first and second kinds, respectively.

As a fluid drop moves forwards, fluid will be pushed outwards and away from the boundary at the front, whereas at the rear fluid will be pulled in toward the drop. This implies an asymmetry in the shape of the plane interface, as sketched in figure 1. Although the interface shape is fundamentally different from that obtained for motion normal to the interface, the results are in many respects qualitatively similar. First, the degree of deformation again increases as the drop moves closer to the interface. Second, the ratio of viscosities, λ, across the plane interface has no effect, to $O(\epsilon^3)$, on the degree of deformation, which is a weak function of the viscosity ratio κ between the drop and the continuous fluid phases. Third, and finally, a density difference across the interface is much more effective than interfacial tension at restricting interface deformation. It is evident that surface tension allows a very broad deformation with small curvature for small values of Φ. In this limit, $\Phi \to 0$ (i.e. in the limit of surface-tension-dominated deformation), however, the solution for $\eta(\mathbf{x}_s)$ in the present development, remains perfectly well-behaved:

$$\eta(\mathbf{x}_s) \approx \frac{3\,\mathrm{Ca}\,A(\kappa)\,x\,d}{\rho^2}\left[1 - \frac{d}{(\rho^2+d^2)^{1/2}}\right]. \quad [59]$$

Unlike the problem of drop motion normal to the interface, $\eta(\mathbf{x}_s)$ does not have log singularity in the limit of $\Phi \to 0$ with $\mathrm{Ca} = O(1)$. In the gravity-dominated limit, i.e. $\Phi \to \infty$, [58] yields

$$\eta(\mathbf{x}_s) \approx \epsilon^3 \frac{9\mathrm{Ca}\,A(\kappa)\,x}{\left(\frac{\rho^2}{d^2}+1\right)^{5/2}\Phi} \quad \text{as } \Phi \to \infty. \quad [60]$$

Thus, the degree of interface distortion is magnified with a decrease in the density difference $\Delta\rho$ between the two continuous fluid phases. Plots which illustrate these conclusions are not included here due to the qualitative similarity in the magnitude of the same effects for the case of motion normal to the interface.

4.3. Lateral migration

Let us now turn to the problem of drop migration in a Newtonian fluid due to flow-induced deformations of the interface and the drop. The basic idea is to consider each of the possible effects, i.e. deformations in the shapes of the interface and the drop, as small corrections to the basic problem of the motion of a *spherical* drop in the vicinity of a *flat* interface. Under these circumstances, as noted in [14], the effects on drop motion are simply additive at first order and can be considered independently. With the first corrections, $\eta^{(1)}(\mathbf{x}_s)$ and $f^{(1)}(\theta, \phi)$, for shapes of the interface and the drop, we can use the reciprocal theorem outlines in section 2 to determine the deformation-induced lateral migration of the drop in translation parallel to the interface. It is evident, from [14], that the evaluation of migration velocity at first order requires only the shape functions, $\eta^{(1)}$ and $f^{(1)}$, the zeroth-order disturbance velocity and stress fields $(\mathbf{u}, \boldsymbol{\sigma})$ obtained in section 4.1 for parallel translation and the velocity and stress fields $(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\sigma}})$ for a "complementary" Stokes problem. The latter is simply the translation of a spherical drop perpendicular to a flat interface—i.e. precisely the solutions already calculated in section 3.1. The order of magnitudes of the integrands in [14] can be obtained using the estimates for $(\mathbf{u}, \boldsymbol{\sigma})$ and $(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\sigma}})$ in each fluid, evaluated on the drop and the interface. A careful examination of the integrals over the undeformed interface, H, and the drop surface, S, in [14] shows that the magnitudes of the migration velocities induced by the interface and the drop deformations are the same and $O(\epsilon^6)$. Thus, the lateral migration velocity at first order can be determined completely by considering the motion of a spherical drop near a slightly deformed interface and the motion of a slightly deformed drop near a flat interface. The situation is somewhat analogous to the lateral migration of a drop in shear flow toward or away from a plane, rigid, wall due to the combined effects of slight non-Newtonian rheology, weak inertia and small deformations in shape (cf. Chan & Leal 1979).

By substitution of the corresponding velocity and stress fields into [14], we determine the dominant contributions to the lateral velocity, $\mathbf{U}_{\rm lm}$:

$$\mathbf{U}_{\rm lm} = \mathbf{U}_{\rm lm}^{\rm D} + \mathbf{U}_{\rm lm}^{\rm I}, \qquad [61]$$

where

$$\mathbf{U}_{\rm lm}^{\rm D} = -\frac{9{\rm Ca}^*\epsilon^6}{1{,}146{,}880}\cdot\left[\frac{\lambda^2(2+3\lambda)}{(1+\lambda)^3}\right]\cdot\left[\frac{(2+3\kappa)^2(16+19\kappa)(54+97\kappa+54\kappa^2)}{(1+\kappa)^5}\right]\cdot\mathbf{e}_z + O(\epsilon^7) \qquad [62]$$

and

$$\mathbf{U}_{\rm lm}^{\rm I} = -\frac{9{\rm Ca}\,\Phi^{1/2}\epsilon^6}{4}\cdot\left[\frac{1}{1+\lambda}\right]\cdot\left[\frac{(2+3\kappa)^2}{(1+\kappa)^2}\right]\cdot\mathbf{e}_z$$

$$\times\int_0^\infty\left[\frac{I_0(\Phi^{1/2}\rho)}{\left(\dfrac{\rho^2}{d^2}+1\right)^3}\int_\rho^\infty\frac{\zeta^2K_1(\Phi^{1/2}\zeta)}{\left(\dfrac{\zeta^2}{d^2}+1\right)^{5/2}}\,{\rm d}\zeta-\frac{K_0(\Phi^{1/2}\rho)}{\left(\dfrac{\rho^2}{d^2}+1\right)^3}\int_0^\rho\frac{\zeta^2I_1(\Phi^{1/2}\zeta)}{\left(\dfrac{\zeta^2}{d^2}+1\right)^{5/2}}\,{\rm d}\zeta\right]{\rm d}\rho + O(\epsilon^7). \qquad [63]$$

It can be seen that both the separate contributions, $\mathbf{U}_{\rm lm}^{\rm D}$ and $\mathbf{U}_{\rm lm}^{\rm I}$, from the deformations of the drop and the interface represent migration *away* from the interface.

The contribution $\mathbf{U}_{\rm lm}^{\rm D}$ to the lateral migration is due completely to the reflected shear flow, $\boldsymbol{\Gamma}\cdot\mathbf{x}$ in [41], by the presence of a flat interface. The interface reflections induce asymmetry in the drop shape, relative to either the direction of the motion or, equivalently, to the interface, which in turn leads to the lateral migration of the drop in a direction normal to the interface. Chaffey *et al.* (1965) and, more recently, Chan & Leal (1979), considered the motion of a deformable drop in a simple shear flow near a *solid* wall using the small deformation theory. They found that the drop would migrate away from the wall, in apparent qualitative agreement with the experimental observations. By comparison, the factor representing dependence on the viscosity ratio κ given by Chan & Leal (1979) is

$$\frac{(16+19\kappa)(54+97\kappa+54\kappa^2)}{(1+\kappa)^3},$$

so that both theories predict a qualitatively similar weak dependence on the viscosity ratio κ. However, the migration velocity, $\mathbf{U}_{\rm lm}^{\rm D}$ given by [62], has a relatively strong dependence on the viscosity ratio λ. In particular, when $\lambda\to 0$, there is no term of $O(\epsilon^6)$ in $\mathbf{U}_{\rm lm}^{\rm D}$. Indeed, as noted in section 4.2, the drop in translation parallel to a free boundary (i.e. $\lambda\to 0$) remains spherical and the lateral position is fixed for all times by its value at some initial instant. It may be noteworthy that the magnitude of the migration velocity predicted by Chan & Leal (1979) for a drop in a simple shear flow is $O(\epsilon^2)$, which is asymptotically large relative to the lateral migration, $\mathbf{U}_{\rm lm}^{\rm D}$ of $O(\epsilon^6)$, for the drop *translation* parallel to a flat interface. This difference can be explained by the weak reflected shear flow of $O(\epsilon^2)$ in [41] induced by the parallel translation.

Let us now examine the contribution $\mathbf{U}_{\rm lm}^{\rm I}$, which is due solely to hydrodynamic interactions between a spherical drop and a deformable interface. The dependences on the viscosity ratios, λ and κ, are qualitatively similar to the contribution of drop deformation, i.e. weak dependence on κ and relatively strong dependence on λ. Indeed, when $\lambda\to\infty$, we see that there is no term of $O(\epsilon^6)$ in $\mathbf{U}_{\rm lm}^{\rm I}$ and the "very viscous" upper fluid has no direct effect on the lateral migration of a spherical drop. A detailed calculation of integrals involved in [63] shows that the contribution $\mathbf{U}_{\rm lm}^{\rm I}$ to the lateral velocity is a monotonically decreasing function of Φ, and decreases asymptotically as $1/\Phi$ in the limit of $\Phi\to\infty$.

5. CONCLUSIONS

The creeping motion of a fluid drop near a fluid–fluid interface has been studied using the standard method of reflections based on the representation of solutions to Stokes' equation in terms of the fundamental singularities for Stokes flow. We have considered an asymptotic limit, $\epsilon \ll 1$, so that the present results are concerned with circumstances in which the interface and drop deformations are both small and dependent solely upon the instantaneous conditions. The small deformation problem is reformulated in terms of equivalent boundary conditions on a *flat* interface and a *spherical* fluid drop; this allows a separation of the arbitrary translation problem, into components parallel and perpendicular to the undeformed interface.

When a fluid drop translates near an interface, we have shown that it is necessary to modify the strength of the Stokeslet and potential dipole singularities at the drop center, as well as add a stresslet and a potential quadrupole through terms of $O(\epsilon^2)$. The theory yields the hydrodynamic drag force on the drop in translation near an interface for arbitrary viscosity ratios which is in very good agreement with exact-solution results in the region of $d \gg 1$. When a fluid drop approaches more closely to the interface, i.e. when $d \approx 1$, an accurate result would require higher-order singularities such as Stokes quadrupoles and potential octupoles at the drop center.

The first corrections for the shapes of the plane interface and the drop have been determined and used to calculate the migration velocity of a drop in translation parallel to an interface. Our calculations have shown that the orders of magnitude of *interface* deformation are $O(\epsilon^2)$ and $O(\epsilon^3)$ for translation perpendicular and parallel to the interface, respectively, while the magnitude of drop deformation is $O(\epsilon^2)$ for an arbitrary translational motion of a fluid droplet. The analysis predicts migration away from the interface with a velocity of $O(\epsilon^6)$. We have also investigated in detail the effects of the viscosity ratios, the capillary numbers and the Bond number on the distortions of the plane interface and drop shapes, and on the lateral migration of the drop.

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